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DYNAMICS OF A SOLID WITH AN ELLIPSOIDAL CAVITY FILLED WITH A MAGNETIC FLUID*

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Equations of motion are derived for a solid with an ellipsoidal cavity filled with a perfect incompressible magnetic fluid. Four first integrals of these equations are shown, their Hamiltonian structure studied and the integrable cases found. A motion of a solid with an ellipsoidal cavity in which the fluid executes a homogeneous vortex motion, was first studied by Lamb, and equations of motions were first derived by Zhukovskii and later by Poincaré, using the group variables. The dynamics of a solid with a cavity filled with magnetic fluid is of interest in connnection with the astrophysical problems arising in the theory of neutron stars and pulsars /1/.

1. Formulation of the problem. Boundary conditions. The dynamics of a perfectly rigid body with an ellipsoidal cavity filled with magnetic fluid, is studied under the following assumptions. The motion of the fluid in the cavity is described by the equations of magnetic hydrodynamics /2,3/

$$\rho\, d\mathbf{v}/dt = -\operatorname{grad} p + (\operatorname{rot} \mathbf{H} \times \mathbf{H})/4\pi \tag{1.1}$$
$$\operatorname{div} \mathbf{v} = 0,\ \ \partial \mathbf{H}/\partial t = \operatorname{rot}(\mathbf{v} \times \mathbf{H}),\ \ \operatorname{div} \mathbf{H} = 0$$

where $\rho = \text{const}$ is the fluid density, v is the velocity vector, p is pressure, and H is the magnetic field intensity vector. The dynamics of the system is studied, as in the classical problem /4/, over the time interval in which the influence of the viscosity of the fluid and its friction against the boundary of the cavity can both be disregarded. We choose a reference system S rigidly attached to the solid, with the origin at the center of mass and the axes parallel to the principal axes of the ellipsoid. In the system S the center O of the ellipsoid has the coordinates r^1, r^2, r^3.

The motion of the fluid in the cavity belongs to the class of motions with homogeneous deformation, which were first studied in magnetic hydrodynamics in /5/. The transformation from the Lagrangian a^k to Eulerian x^i coordinates has the form

$$x^i = F_k{}^i\,(t)\, a^k + (Q_1)_k{}^i\, r^k,\ \ F = Q_1 D Q_2,\ D = \operatorname{diag}\{d_1,\ \ d_2,\ \ d_3\} \tag{1.2}$$

where Q_1 and Q_2 are orthogonal matrices, d_1, d_2, d_3 are the ellipsoid semiaxes, and the Lagrangian coordinates a^k pass through the unit sphere $(a^1)^2 + (a^2)^2 + (a^3)^2 \leqslant 1$. The repeated indices denote summation everywhere. The magnetic field intensity with components H^i in the cavity, has the form

$$H^i = F_k{}^i h_j{}^k a^j \tag{1.3}$$

at the point (1.2), where $\| h_j{}^k \|$ is a constant, skew symmetric matrix.

The electromagnetic field has a discontinuity at the surface of the ellipsoid. Outside the ellipsoid the electromagnetic field is absent, since the internal electromagnetic field is fully screened by the surface current and surface charge. We shall show that all necessary boundary conditions hold at the discontinuity. Let $H_n, H_\tau, E_n, E_\tau, v_n, v_\tau$ be the normal and tangential components of the magnetic field, electric field and velocity of the fluid at the ellipsoid surface. The conditions at the discontinuity have the following form in the magnetic hydrodynamics /2,3/ (heat conductivity is neglected)

$$\{E_\tau\} = 0,\quad \{E_n\} = 4\pi\theta,\quad \{H_n\} = 0,\quad \{H_\tau\} = 4\pi c^{-1}(\mathbf{i} \times \mathbf{n}) \tag{1.4}$$

$$\{\rho v_n\} = 0,\ \ \{s_n - (P\cdot\mathbf{v})\cdot\mathbf{n} + \rho v_n(\varepsilon + v^2/2)\} = 0 \tag{1.5}$$

$$\{\rho v_n \mathbf{v} - P\cdot\mathbf{n} - T\cdot\mathbf{n}\} = 0 \tag{1.6}$$

$$\mathbf{s} = c(\mathbf{E} \times \mathbf{H})/(4\pi),\ P_{ij} = -p\delta_{ij},\quad T_{ij} = (H_i H_j - H^2\delta_{ij}/2)/(4\pi)$$

Here θ is the surface charge, $\mathbf{i}$ is the surface current, $\mathbf{n}$ is the vector normal to the surface of the discontinuity, $\mathbf{s}$ is the electromagnetic energy flux density vector, P and T are matrices with components P_{ij} and T_{ij}, and ε is the density of internal energy of the fluid.

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By virtue of (1.2) and (1.3) we have $v_n = 0$, $H_n = 0$. Within the approximation of the magnetic hydrodynamics $\mathbf{E} = (\mathbf{v} \times \mathbf{H})/c$, therefore $\mathbf{E}_\tau = 0$. Consequently the condition (1.4) hold and determine the surface current and the surface charge. The conditions (1.5) hold by virtue of $v_n = 0$, $s_n = 0$. Conditions (1.6) lead, by virtue of $v_n = 0, H_n = 0$, to condition $\{p + H^2/8\pi\} = 0$, which determines the pressure from the side of the elastic shell, since it is assumed that $\mathbf{H} = 0$ outside the cavity and therefore it also holds in the case of a perfectly rigid body.

2. Equations of motion. The equations of magnetic hydrodynamics (1.1) and law of conservation of total mommentum of impulse together constitute the equations of motion of a solid with a cavity. Let us introduce the notation

$$Q_1^{\cdot} = Q_1 A,\ Q_2^{\cdot} = -BQ_2 \tag{2.1}$$

and use the known property of isomorphism of the vectors with components v^i in R^3 and skew symmetric (3×3)-matrices with components V_{jk}:

$$v^i \to V_{jk} = -v^i \varepsilon_{ijk} \tag{2.2}$$

under which the vector product $x \times y$ becomes a commutator of the matrices $[X, Y] = XY - YX$. After this isomorphism, the vectors with components A^i, B^i, $i = 1, 2, 3$ correspond to the skew symmetric matrices A and B.

The moment of impulse of the fluid within the cavity (relative to the point O) has the form (the integral is taken everywhere over the volume of the cavity)

$$M_0^i = \rho \int (\mathbf{x} \times \mathbf{v})^i dx^1 dx^2 dx^3 = -\frac{1}{2} \varepsilon_{ijk} M_{jk} + (Q_1)_j^i I_{jk}^0 A^k$$

$$M = m_1 (F^{\cdot} F^t - F F^{\cdot t}) = m_1 Q_1 (D^2 A + A D^2 - 2DBD) Q_2$$

$$I_{jk}^0 = m((r^l)^2 \delta_{jk} - r^j r^k),\ m = 4\pi d_1 d_2 d_3 / 3,\ m_1 = m/5$$

where m is the total mass of the fluid, index t denotes transposition and M_{jk} are components of the matrix M. The vector $\overline{\mathbf{M}} = m_1 \mathbf{M}$ of total moment of impulse of the body and the fluid has the following components in the reference system S:

$$M^i = I_{ik} A^k - 2B^i d_j d_k,\ i, j, k = 1, 2, 3 \tag{2.3}$$

$$I_{ik} = b_i \delta_i^k + I_{ik}^0 m_1^{-1} + I_{ik}^1 m_1^{-1},\quad b_i = d_j^2 + d_k^2$$

where I_{ik}^1 is the inertia tensor of the solid in the system S. The law of conservation of total momentum of impulse has the form

$$\mathbf{M}^{\cdot} = \mathbf{M} \times \mathbf{A} \tag{2.4}$$

The last three equations of magnetic hydrodynamics (1.1) hold identically by virtue of the definitions (1.2) and (1.3). Passing to the problem of transforming the first equation of (1.1) we note, that in case of the motions with homogeneous deformation the pressure p is a quadratic function of the coordinates

$$p = p_0(t) + p_{ij}(t) a^i a^j + p_i(t) a^i$$

where $p_{ij}(t)$ are components of symmetric matrix $P_0(t)$. Substituting the above expression for pressure and the formulas (1.2) and (1.3) into the first equation of (1.1), we find that the equation is equivalent to the following matrix and vector equation:

$$\rho F^{\cdot\cdot} = -(F^{-1})^t P_0 + ((F^{-1})^t h F^t F h + F h^2)/(4\pi) \tag{2.5}$$

$$p_i(t) = -F_i^k (Q_1^{\cdot})_l^k r^l$$

Let us write $K_0 = F^{\cdot t} F - F^t F^{\cdot}$. Clearly $K_0^{\cdot} = F^{\cdot\cdot t} F - F^t F^{\cdot\cdot}$ represents the antisymmetric part of the matrix $F^t F^{\cdot\cdot}$. The symmetric part of this matrix defines the matrix $P_0(t)$

$$2P_0 = -\rho (F^t F^{\cdot\cdot} + F^{\cdot\cdot t} F) + (2\pi)^{-1} h F^t F h + (4\pi)^{-1} (F^t F h^2 + h^2 F^t F) \tag{2.6}$$

By virtue of (2.5) we have

$$\rho K_0^{\cdot} = (4\pi)^{-1} (h^2 F^t F - F^t F h^2) \tag{2.7}$$

Using the definition (2.1) we obtain

$$K_0 = Q_2^t K Q_2,\ K = D^2 B + B D^2 - 2DAD,\ F^t F = Q_2^t D^2 Q_2 \tag{2.8}$$

Using the above formulas we transform the equation (2.7) to the equivalent form

$$K^{\cdot} = [K, B] + \varkappa [Q_2 h^2 Q_2^t,\ D^2],\ \varkappa = (4\pi\rho)^{-1} \tag{2.9}$$

where the square brackets denote the commutator of the matrices. Writing $u = Q_2 h Q_2^t$, and using

(2.1) we obtain

$$u^{\cdot} = [u, B],\ [u^2, D^2] = [u, uD^2 + D^2u] \tag{2.10}$$

After the isomorphism (2.2), the vector $\mathbf{u}$ with components u^1, u^2, u^3; corresponds to the skew symmetric matrix u and vectors $\mathbf{K}$ and $\mathbf{w}$ with components

$$K^i = b_i B^i - 2A^i d_j d_k,\ w^i = b_i u^i,\ i, j, k = 1, 2, 3 \tag{2.11}$$

correspond to the matrices K and $uD^2 + D^2u$ (In the last expression summation over i is omitted). Using vector notations (2.3) and (2.11) we find that the equations (2.4), (2.9) and (2.10) define a complete system of equations describing the dynamics of a solid with an ellipsoidal cavity filled with magnetic fluid

$$\mathbf{M}^{\cdot} = \mathbf{M} \times \mathbf{A}, \quad \mathbf{K}^{\cdot} = \mathbf{K} \times \mathbf{B} + \varkappa \mathbf{u} \times \mathbf{w}, \quad \mathbf{u}^{\cdot} = \mathbf{u} \times \mathbf{B} \tag{2.12}$$

Equations (2.12) determine fully the dependence of the matrix F on time, therefore the equation (2.6) and second equation of (2.5) enable us to find the matrix $P_0(t)$ and the coefficients $p_i(t)$. i.e. to compute the pressure within the fluid (with the accuracy of up to the nonessential addition constant). Equations (2.12) represent a generalization of the classical equations of motion of a body with an ellipsoidal cavity filled with a perfect incompressible fluid, and have been derived here for the first time. The classical case corresponds to the absence of magnetic field and is obtained from (2.12) by setting $\mathbf{u} = 0$.

3. First integrals of the system. Integrable cases. The most important integral of the system (2.12) is the total energy integral E, which consists of the kinetic energy of the fluid E_1, internal magnetic field energy E_2 and kinetic energy of rotation of the solid E_3

$$E_1 = \int \frac{\rho v^2}{2} dx^1 dx^2 dx^3 = \frac{1}{2} m_1 \operatorname{tr}(F^{\cdot}F^{\cdot t}) + \frac{1}{2} I_{ik}^0 A^i A^k \tag{3.1}$$

$$E_2 = \int \frac{H^2}{8\pi} dx^1 dx^2 dx^3 = \frac{1}{30} \operatorname{tr}(h^t F^t F h)\, d_1 d_2 d_3$$

$$E_3 = \frac{1}{2} I_{ij}{}^1 A^i A^j, \qquad E = E_1 + E_2 + E_3$$

Using the notation of Sect.2 in the above formulas, we obtain

$$2H = 2E/m_1 = (\mathbf{M}, \mathbf{A}) + (\mathbf{K}, \mathbf{B}) + \varkappa(\mathbf{u}, \mathbf{w}) = \tag{3.2}$$

$$I_{ik} A^i A^k - 4A^i B^i d_j d_k + b_i (B^i)^2 + \varkappa b_i (u^i)^2$$

$$b_i = d_j^2 + d_k^2,\ i, j, k = 1, 2, 3$$

Clearly $M^i = \partial H/\partial A^i$, $K^i = \partial H/\partial B^i$, $\varkappa w^i = \partial H/\partial u^i$. We can confirm by direct substitution that the function $J_1 = H$ represents the first integral of the system (2.12). The system can also have the following three first integrals:

$$J_2 = (\mathbf{M}, \mathbf{M}), \qquad J_3 = (\mathbf{u}, \mathbf{u}), \qquad J_4 = (\mathbf{K}, \mathbf{u}) \tag{3.3}$$

Here J_2 is a square of the total angular momentum, J_3 is a square of the magnetic field intensity vector $\mathbf{h}$ in Lagrangian coordinates, and J_4 is a scalar product of the fluid vorticity vector and vector $\mathbf{h}$, all with the accuracy of up to a multiplier. The combined level of the three integrals (3.3) defines a six-dimensional manifold $V^6 = T(S^2) \times S^2$ represented by a product of a bundle tangent to a two-dimensional sphere, and the two-dimensional sphere S^2.

The system (2.12) represents a special case of the Euler equations /6/ on the Lie algebra A, the latter being a sum of the Lie algebra of the group E_3 of motions of the three-dimensional space and Lie algebra SO /3/. It follows therefore that a simplectic structure is defined on the manifold V^6 in the standard manner /6/; the system (2.12) on it is Hamiltonian, with the Hamiltonian H.

In the case of a spherical cavity $(d_1 = d_2 = d_3)$ the magnetic field does not have any effect on the dynamics of the system, and equations (2.12) reduce to the usual Euler equations of motion for some effective solid. In the case of axial symmetry of the solid and cavity $d_1 = d_2$, $r^i = (0, 0, r^3)$, $I_{ik} = I_i \delta_i^k$, $I_1 = I_2$, the equations (2.12) have an additional first integral $J_5 = M^3 + K^3$ and invariants relative to the group of simultaneous turns in the planes (M^1, M^2), (K^1, K^2) and (u^1, u^2). Therefore the system (2.12) lies on the combined level of the first integrals (3.3) and J_5 and reduces, after the factorization on the group shown above, to a Hamiltonian system on a four-dimensional manifold which is, in general, nonintegrable.

Let us consider an important case of the zero total angular momentum of the system, $J_2 = 0$. We assume that the center of mass lies at the center of the ellipsoid $(r^i = 0)$ and the inertia tensor of the solid is diagonal $I_{ik}{}^1 = I_i \delta_i^k$. Then by virtue of (2.3) we have

$$A^i = 2B^i d_j d_k/(b_i + I_i m_1^{-1})$$

and the system (2.12) becomes

$$\begin{aligned}&\overline{\mathbf{K}}^{\cdot} = \overline{\mathbf{K}} \times \mathbf{B} + \varkappa \mathbf{u} \times \mathbf{w}, \quad \mathbf{u}^{\cdot} = \mathbf{u} \times \mathbf{B}, \quad w^i = \partial \overline{H}/\partial u_i \\ &B^i = \partial \overline{H}/\partial \overline{K}_i, \quad 2H = c_i^{-1}\overline{K}^2 + \varkappa b_i u_i^2, \\ &c_i = b_i - 4d_j^2 d_k^2 (b_i + I_i m_1^{-1})^{-1}\end{aligned} \tag{3.4}$$

Equations (3.4) are analogous to the classical Kirchhoff equations of motion of a solid with three symmetry planes in a perfect incompressible fluid. From the theory of Kirchhoff equations /7/ we know that the system (3.4) has, provided that the Klebsch condition

$$c_1(b_2 - b_3) + c_2(b_3 - b_1) + c_3(b_1 - b_2) = 0 \tag{3.5}$$

holds, and an additional first integral

$$J = \overline{K}_1^2 + \overline{K}_2^2 + \overline{K}_3^2 + \varkappa c_2(b_1 - b_3)u_1^2 + \varkappa c_1(b_2 - b_3)u_2^2 \tag{3.6}$$

is therefore fully integrable.

We shall show that a two-parameter family of values of inertia tensor I_i of the solid exists for any values of the semiaxes of the ellipsoidal cavity d_1, d_2, d_3 for which the relation (3.5) holds, i.e. the dynamic system is integrable at the level $J_2 = 0$. Let $d_3 > d_1 > d_2$; and introduce the following notation:

$$\begin{aligned}&\beta_i = 1 + I_i m_1^{-1} b_i^{-1} > 1, \quad x_1 = d_1 d_3^{-1}, \quad x_2 = d_2 d_3^{-1}, \quad x_2 < x_1 < 1 \\ &\alpha_1 = 2x_2(1 + x_1^2)^{-1}, \quad \alpha_2 = 2x_1(1 + x_2^2)^{-1}, \quad \alpha_3 = 2x_1 x_2 (x_1^2 + x_2^2)^{-1}\end{aligned} \tag{3.7}$$

From (3.4) we obtain $c_i = b_i(1 - \alpha_i^2 \beta_i^{-1})$. After substituting (3.7) into (3.5) and transforming, we obtain

$$\frac{x_1^2(1 - x_1^2)}{\beta_2(1 + x_1^2)} + \frac{x_2^2(x_2^2 - 1)}{\beta_1(x_2^2 + 1)} + \frac{x_1^2 x_2^2(x_1^2 - x_2^2)}{\beta_3(x_1^2 + x_2^2)} = 0 \tag{3.8}$$

From (3.8) with $0 < x_2 < x_1 < 1$ and two arbitrary parameters $\beta_2, \beta_3 > 0$ we find, that $\beta_1 > 0$. Solutions of (3.8) admit the transformation $\beta_i \to L\beta_i$, therefore at sufficiently large L we obtain a two-parameter family of solutions with $\overline{\beta}_i = L\beta_i > 1$. Next we find from (3.7) the corresponding components of the inertia tensor I_i of the solid. In particular, (3.8) has solutions for which $x_1 \approx x_2 \approx 1$ and $\beta_1 \approx \beta_2 \approx \beta_3$. In this case the necessary conditions $I_i < I_j + I_l$. also hold for large L.

The known integrable S.A. Chaplygin case for the Kirchhoff equations also leads to an integrable case of the system under consideration. Equations (3.4) with conditions

$$c_1 = c_2 = 2c_3, \quad b_3 = (b_1 + b_2)/2, \quad J_4 = 0 \tag{3.9}$$

have an additional first integral

$$J_5 = ((K_2^2 - K_1^2)c_3^{-1} + \varkappa(d_1^2 - d_2^2)u_3^2)^2 + 4c_3^{-2}K_1^2 K_2^2$$

and are therefore fully integrable. Substituting the expressions (3.4) and (3.7) we can show that (3.9) have a three-parameter family of solutions d_i, I_k satisfying the necessary conditions.

4. Astrophysical applications. The class of solutions constructed here models the dynamics of rotation of such astrophysical objects as neutron stars and pulsars. According to modern ideas /1/, the neutron stars have a fluid center within a solid shell which is strongly conducting and has a strong magnetic field frozen into it. Electromagnetic radiation emitted by such objects over a long period of time is strictly periodic, therefore the problem of existence of periodic solutions (for which the matrices $Q_1(t), Q_2(t)$ are periodic functions of t and have the same period), is of importance.

It is well known that such solutions correspond to the closed trajectories of the system (2.12) (the converse is not true, since the equations (2.12) describe the variation in the physical quantities relative to the reference system S attached to the rotating body). Solutions which have minimum total energy at the given level of the integrals (3.3), are of special importance. Such solutions exist by virtue of the positive definiteness of the Hamiltonian H at every level of the integrals (3.3), and correspond to certain stationary points of the system (2.12). The solutions are stable, and by virtue of existence of numerous mechanisms of the energy loss, rotation of the real objects must tend, with time, namely to such solutions. The following conditions hold at the stationary points of the system (2.12):

$$M^i = \lambda A^i, \quad u^i = \mu B^i, \quad K^i = \varkappa\mu w^i + \gamma B^i \tag{4.1}$$

Using the formulas (2.3) and (2.11) we obtain three equations for determining the coefficients λ, μ, γ

$$(b_i(1-\varkappa\mu^2)-\gamma)(b_i+I_im_1^{-1}-\lambda)=4d_j^2d_k^2 \quad (i, j, k = 1, 2, 3) \tag{4.2}$$

By virtue of (2.1), the matrices $Q_1(t)$ and $Q_2(t)$ in solutions corresponding to the stationary points (4.1) and (4.2) (and in particular to the stable points of the minima of H) describe periodic rotations with periods T_1 and T_2. If the periods T_1 and T_2 are incommensurable, then the frozen-in magnetic field H intensity vector (1.3) will, for every point of the cavity in the Eulerian coordinates x^j, vary quasiperiodically and fill densely everywhere a two-dimensional surface obtained by rotating an ellipse about its fixed axis (if this surface has no self-intersections, then it is a torus). If the periods T_1 and T_2 are commensurable, then the solution is strictly periodic and vector H describes a closed (possibly self-intersecting) curve. It must be stressed that in the case of fluid rotating as a solid and possessing minimum energy at the given level of the moment impulse integral, the frozen-in magnetic field vector always describes (in Eulerian coordinates) a plane circumference.

In the case of zero total moment of impulse the equations (2.12) reduce, as was shown before, to the Kirchhoff equations. Therefore form the results of /8/ it follows that when $J_2=0$, then at every level of the integrals J_3, J_4 and $H=J_1>E(J_3, J_4)$ there exist at least two closed trajectories of the system (2.12)−(3.4). The case $J_2=0$ models the dynamics of rotation of the neutron stars grown from massive objects with a small momentum. For this reason, at the final outcome of their evolution, i.e. when a neutron star is formed, the moment of impulse of the shell and the fluid center compensate each other.

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